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# Specific heat of quantum elastic systems pinned by disorder

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**Abstract.** We present the detailed study of the thermodynamics of vibrational modes in disordered elastic systems such as the Bragg glass phase of lattices pinned by quenched impurities. Our study and our results are valid within the (mean field) replica Gaussian variational method. We obtain an expression for the internal energy in the quantum regime as a function of the saddle point solution, which is then expanded in powers of $\hbar$ at low temperature $T$. In the calculation of the specific heat $C_v$ a non trivial cancellation of the term linear in $T$ occurs, explicitly checked to second order in $\hbar$. The final result is $C_v \propto T^3$ at low temperatures in dimension three and two. The prefactor is controlled by the pinning length. This result is discussed in connection with other analytical or numerical studies.

**PACS.** 75.10.Nr Spin-glass and other random models – 75.40.Cx Static properties (order parameter, static susceptibility, heat capacities, critical exponents, etc.) – 65.60.+a Thermal properties of amorphous solids and glasses: heat capacity, thermal expansion, etc.

## 1 Introduction

Despite progress in analytical solutions of models of glasses in solvable limits such as mean field, there is at present little known information about detailed thermodynamic properties of experimental interest such as the specific heat. In particular an outstanding question is its linear behavior at low temperature [1]. Such behavior was measured in a large variety of experimental systems including amorphous solids, disordered crystals and spin glasses [2–4]. A phenomenological approach, based on the existence of two level systems [5] was proposed leading to a linear temperature dependence of the specific heat. Despite the success of such prediction for many experimental systems, the question of the validity and applicability of such arguments is still under debate. In particular, for many glassy systems, the microscopic origin of the assumed two level systems remains unclear.

Computing the specific heat from a microscopic model is of course an extremely difficult problem for a disordered system. However, mean field theory of quantum spin glass, have been studied [6,7] leading to results in agreement with the linear dependence of the specific heat. Nevertheless the validity of the linear temperature dependence is being challenged [8,9]. Another important class of glasses to which these mean field methods have been applied with success to compute correlation functions [10], consists in disordered elastic systems. Such systems cover a wide range of experimental situations such as charge density waves [11], electron glasses [12,13], and flux lattices [14–17], for which the quantum limit is of interest. Thus both from the experimental and theoretical point of view such disordered elastic systems are ideal to address this important question of the behavior of the specific heat.

This is the question that we address in the present paper. We consider these systems in the elastic limit whenever topological defects can be neglected. This is the case within the Bragg glass phase which was studied previously in both classical [10] and quantum [18] limit using the Gaussian variational approximation [19,10,18] to the replicated Hamiltonian. We use in this paper the same variational approach to compute the specific heat for these systems. In two previous short papers [20,21], we have summarized the result of the calculation of the specific heat for these systems in classical and quantum limits, as well as the applications to superconductors. In the present paper we give a detailed exposition of the method. Indeed two important questions must be addressed. In order to obtain the correct temperature dependence, it is necessary to push the understanding of the structure of the mean field solution beyond what has been achieved in reference [18]. In addition obtaining the amplitude requires performing a semiclassical expansion. Besides, for systems in dimension $d < 2$, two saddle points solutions for the

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variational equations exist, one coming from the thermodynamics and one from the so-called marginality condition. We discuss the respective roles of these solutions.

The outline of the paper is as follows. In Section 2 we formulate the problem of the calculation of the internal energy for an elastic system with quenched disorder. In Section 3 we recall the replica variational method and obtain a compact expression for the internal energy in terms of the saddle point solution. This expression being very hard to compute analytically we present a systematic expansion in powers of $\hbar$, given that the specific heat is a function $C_v(\beta\hbar, \hbar)$. In Section 4 we compute the first two terms of this expansion, the details being given in Appendix A and Appendix B. The result is a cancellation of the linear term, resulting in $C_v \sim T^3$.

## 2 Specific heat computation using functional integral

### 2.1 Model

We consider a collection of interacting quantum objects of mass $M$ whose position variables are denoted $u_\alpha(R_i, \tau)$. The equilibrium positions $R_i$ in the absence of any fluctuations form a perfect lattice of spacing $a$. Interactions result in an elastic tensor $\Phi_{\alpha,\beta}(q)$ which describes the energy associated to small displacements, the phonon degrees of freedom. Impurity disorder is modelled by a $\tau$ independent Gaussian random potential $U(x)$ directly coupled to the local density $\rho(x) = \sum_i \delta(x - R_i - u(R_i, \tau))$. We will describe systems in the weak disorder regime $a/R_a \ll 1$ where $R_a$ is the translational correlation length, e.g. in a Bragg glass phase where the condition $|u_\alpha(R_i, \tau) - u_\alpha(R_i + a, \tau)| \ll a$ holds, no dislocation being present. The system at equilibrium at temperature $T = 1/\beta$ is described by the partition function $Z = \mathrm{Tr} e^{-\beta H[\Pi,u]} = \int DuD\Pi e^{-S/\hbar}$ with the Hamiltonian $H[\Pi, u] = H_{\rm ph}[\Pi, u] + H_{\rm dis}[\Pi, u]$:

$$H_{\rm ph}[\Pi, u] = \frac{1}{2}\int_q \frac{\Pi(q)^2}{M} + \sum_{\alpha,\beta} u_\alpha(q)\Phi_{\alpha,\beta}(q)u_\beta(-q)$$
$$H_{\rm dis}[\Pi, u] = \int d^d x U(x)\rho(x, u(x)) \quad (1)$$

and its associated Euclidean quantum action in imaginary time $\tau$

$$S[\Pi, u] = -\int_0^{\beta\hbar} d\tau \int_q i\Pi_\alpha(q, \tau)\partial_\tau u_\alpha(q, \tau) + H[\Pi, u] \quad (2)$$

where $u(q, \tau)$ and its conjugated momentum $\Pi(q, \tau)$ satisfy periodic boundary conditions, of period $\beta\hbar$, along the $\tau$ axis. One denotes by $\int_q \equiv \int_{BZ} \frac{d^d q}{(2\pi)^d}$ integration on the first Brillouin zone. For simplicity we illustrate the calculation on a isotropic system with $\Phi_{\alpha,\beta}(q) = cq^2\delta_{\alpha\beta}$ and denote disorder correlations $\overline{U(x)U(x')} = \Delta(x - x')$, with $\overline{U(x)} = 0$.

### 2.2 Pure phonons

We first consider the case of pure phonons, described by a purely isotropic elastic Hamiltonian, i.e. (1) with $U = 0$:

$$H_{\rm ph}[\Pi, u] = \frac{1}{2}\int_q \frac{\Pi(q)^2}{M} + cq^2 u(q)u^*(q). \quad (3)$$

One computes the specific heat per unit volume $C_v(T)$ of this system (3) using the functional integral in imaginary time (2), $C_v(T)$ being defined by

$$C_v(T) = \frac{1}{\Omega}\frac{\partial\langle H_{\rm ph}[\Pi, u]\rangle}{\partial T} \quad (4)$$
$$\langle H_{\rm ph}[\Pi, u]\rangle = \frac{1}{Z_{\rm ph}}\mathrm{Tr}[H_{\rm ph}[\Pi, u]e^{-\beta H_{\rm ph}[\Pi,u]}]$$

with $\Omega$ the volume of the system. $H_{\rm ph}[\Pi, u]$ being independent of the imaginary time $\tau$, $H_{\rm ph}[\Pi, u](\tau) = e^{\tau H_{\rm ph}[\Pi,u]}H_{\rm ph} \times e^{-\tau H_{\rm ph}[\Pi,u]} = H_{\rm ph}[\Pi, u]$, one writes $\langle H_{\rm ph}[\Pi, u]\rangle$ as

$$\langle H_{\rm ph}[\Pi, u]\rangle = Z_{\rm ph}^{-1}\int \mathcal{D}\Pi\mathcal{D}u e^{-\frac{S_{\rm ph}}{\hbar}}\frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau H_{\rm ph}[\Pi, u](\tau) \quad (5)$$

where $S_{\rm ph}$ is the Euclidean quantum action (2) corresponding to (3). One introduces the Fourier transforms (w.r.t. imaginary time variable) of the fields in terms of Matsubara frequencies $\omega_n = 2\pi n/(\beta\hbar)$:

$$u(q, \omega_n) = \int_0^{\beta\hbar} d\tau e^{i\omega_n\tau}u(q, \tau) \quad (6)$$

and similarly for $\Pi(q, \omega_n)$. After integration over the field $\Pi$ in (5), one obtains by performing the change of variable $\Pi(q, \tau) \to \Pi(q, \tau) + iM\partial_\tau u(q, \tau))$:

$$\frac{1}{\Omega}\langle H_{\rm ph}[\Pi, u]\rangle = \frac{\hbar}{\beta\hbar}\int_q \sum_n \frac{cq^2}{M\omega_n^2 + cq^2}. \quad (7)$$

Performing the sum over Matsubara frequencies, e.g. by using the spectral representation of the 2-point Green function $1/(M\omega_n^2 + cq^2)$ one obtains

$$\frac{1}{\Omega}\langle H_{\rm ph}[\Pi, u]\rangle = H_{\rm ph}^{\rm ZP} + \hbar v\int_q |q| f_B(v|q|) \quad (8)$$

where $H_{\rm ph}^{\rm ZP}$ is the (temperature independent) zero-point energy, $f_B(x) = (e^{\beta\hbar x} - 1)^{-1}$ is the Bose factor and $v = \sqrt{c/M}$ the pure phonon velocity. Computing the specific heat (5) from (8) one recovers the Debye law for pure phonons

$$C_{v\rm Debye}(T) = A_d\left(\frac{T}{\hbar v}\right)^d + \mathcal{O}\left(T^{d+1}\right) \quad (9)$$

where $A_d = \Gamma(2 + d)\zeta(1 + d)K_d$, with $\zeta(x)$ the Riemann zeta function, $K_d = S_d/(2\pi)^d$, $S_d$ being the volume of the $d$-dimensional unit sphere, e.g. $A_3 = 2\pi^2/15$. This method using a functional integral formulation (5) is thus a very convenient way to compute the specific heat of elastic systems.

### 2.3 Disordered case: general expression

We now extend this analysis to the disordered case (1) and obtain the analogous formula of (7) for the disordered average internal energy. We remind the expression of the disordered elastic Hamiltonian

$$H[\Pi,u] = H_{\rm ph}[\Pi,u] + \int_x U(x)\rho(u(x),x) \qquad (10)$$
$$\overline{U(x)U(x')} = \Delta(x-x')$$

and denote by $P(U(x))$ the Gaussian distribution of the disorder. As $\rho(u(x),x)$ depends only on the field $u(x)$ and the disorder is $\tau-$*independent*, we compute $\langle H[\Pi,u]\rangle$ in one realization of the disordered potential in the same way as for the pure case (5) with the substitution $H_{\rm ph}[\Pi,u] \to H[\Pi,u]$ (11). To compute the average over the disorder, we introduce replicas to deal with the denominator in (5), using $Z^{-1} = \lim_{k\to 0} Z^{k-1}$:

$$\overline{\langle H\rangle} \equiv \overline{\langle H[\Pi,u]\rangle} =$$
$$\lim_{k\to 0}\int \mathcal{D}U P(U(x)) \int \mathcal{D}\Pi^a \mathcal{D}u^a \frac{1}{\beta\hbar}\int_0^{\beta\hbar} d\tau H\left[\Pi^{a_1},u^{a_1}\right](\tau) \times e^{-\frac{1}{\hbar}\sum_a S[\Pi^a,u^a]} \qquad (11)$$

where $S[\Pi^a,u^a]$ is the action defined by (2) and $a=1,..k$ is a replica index, $a_1$ being one of these replicas, and $\mathcal{D}\Pi^a \equiv \mathcal{D}\Pi^1...\Pi^k$ and similarly for $\mathcal{D}u^a$. As for the pure case, the Gaussian integrals over the fields $\Pi^a$ is straightforwardly computed (performing the same change of variable). To treat the integral over the disorder $U(x)$, one uses the identity for a Gaussian variables with correlations $\Delta(x-x')$ (11)

$$\overline{f:U\exp\left(-f:U\right)} = -f:\Delta:f\exp\left(\frac{1}{2}f:\Delta:f\right) \qquad (12)$$

for any vector $f\equiv f(x)$ and where $f:U=\int_x f(x)U(x)$ and $f:\Delta:f=\int_{xx'} f(x)\Delta(x-x')f(x')$. Applying this formula (12) to (11) with $f(x)=\frac{1}{\hbar}\sum_a\int_0^{\beta\hbar}d\tau\rho(u^a(x,\tau),x)$ (11), one obtains the *exact* formula for the internal energy:

$$\overline{\langle H\rangle} = \frac{\hbar}{\beta\hbar}\sum_n\int_q \frac{1}{2} + \left(cq^2 - M\omega_n^2\right)\tilde{G}(q,\omega_n)$$
$$-\lim_{k\to 0}\frac{1}{\beta\hbar^2}\frac{1}{k}\sum_{a,b=1}^{k}\int_0^{\beta\hbar}d\tau\int_0^{\beta\hbar}d\tau'\int_x\int_{x'}\Delta(x-x')$$
$$\times\left\langle\rho(u^a(x,\tau),x)\rho\left(u^b\left(x',\tau'\right),x'\right)\right\rangle_{\rm rep} \qquad (13)$$

where $\tilde{G}(q,\omega_n)\equiv\langle u^a(q,\omega_n)u^{a*}(q,\omega_n)\rangle_{\rm rep}$ and the averaged values in (15) $\langle ...\rangle_{\rm rep}=\int\mathcal{D}u^a...e^{-\frac{S^{\rm rep}}{\hbar}}$ are computed with the replicated action given below (16). Writing the density $\rho(u(x),x)$ as [10]

$$\rho(u(x),x) = \rho_0\left(1-\nabla\cdot u(x)+\sum_{K\neq 0}e^{iK\cdot(x-u(x))}\right) \qquad (14)$$

with $K$ the reciprocal lattice vectors of the perfect lattice and $\rho_0\propto a^{-2}$ the average density, the second term of (13) can be written as, discarding irrelevant terms:

$$\overline{\langle H\rangle} = \frac{\hbar}{\beta\hbar}\sum_n\int_q\frac{1}{2}+\left(cq^2-M\omega_n^2\right)\tilde{G}(q,\omega_n)$$
$$-\lim_{k\to 0}\frac{1}{k}\frac{1}{\beta\hbar^2}\int d^dx d\tau d\tau'\sum_{a,b=1}^{k}\langle R(u_a(x,\tau)-u_b(x,\tau'))\rangle_{\rm rep}$$
$$R(u) = \rho_0^2\sum_K \Delta_K\cos(K\cdot u). \qquad (15)$$

The same kind of manipulations [10] lead to the following replicated action

$$S^{\rm rep}[u] = \int_q\sum_{n,a}\frac{1}{2\beta\hbar}\left(cq^2+M\omega_n^2\right)u^a(q,\omega_n)u^a(-q,-\omega_n)$$
$$-\frac{1}{2\hbar}\int d^dx d\tau d\tau'\sum_{ab}R(u_a(x,\tau)-u_b(x,\tau')). \qquad (16)$$

In these expressions (15, 16), $\Delta_K=\int d^dx e^{iK\cdot x}\Delta(x)$ denote the harmonics of the disorder correlator at the reciprocal lattice vectors $K$. The exact expression (15) is the starting point of our computation of the specific heat.

## 3 Variational computation

Given the complexity of the replicated action (16), we study it within the Gaussian variational approximation [19,10]. The use of such a method for the calculation of the specific heat necessitates to compute the full temperature and $\hbar$ dependence of the saddle point solution. Before establishing the formulas for the specific heat in Section. 3.3, we present the full variational solution in Sections. 3.1 and 3.2.

### 3.1 Saddle point equations

The variational method [19,10] is implemented by choosing a Gaussian variational action $S_0$, parameterized by a $k\times k$ matrix in replica space $G_{ab}^{-1}(q,\omega_n)$:

$$S_0 = \frac{1}{2\beta\hbar}\int_q\sum_{a,b}G_{ab}^{-1}(q,\omega_n)u^a(q,\omega_n)u^b(-q,-\omega_n)$$
$$G_{ab}^{-1}(q,\omega_n) = cq^2\delta_{ab}-\sigma_{ab} \qquad (17)$$

which minimizes the variational free energy

$$F^{\rm var} = F_0 + \frac{1}{\beta\hbar}\langle S^{\rm rep}-S_0\rangle_{S_0} \qquad (18)$$
$$F_0 = \frac{1}{\beta}\int_q\sum_n(\ln G)_{aa}(q,\omega_n).$$

The disordered term in $S^{\rm rep}$ (16) being purely local in space, but bi-local in time the self energy does not depend on $q$ and depends only on $\omega_n$, thus $\sigma_{ab}\equiv\sigma_{ab}(\omega_n)$.

However, the disorder potential being $\tau$-independent, non diagonal terms (in replica space) such that $\sigma_{a\neq b}$ do exist only [18] for $\omega_n = 0$. Indeed, before averaging over the disorder, different replicas are independent:

$$G_{a\neq b,U} = \left\langle u^a(x,\tau)u^b(0,0)\right\rangle = \langle u^a(x,\tau)\rangle \left\langle u^b(0,0)\right\rangle = \langle u^a(x,0)\rangle \left\langle u^b(0,0)\right\rangle . \quad (19)$$

In the limit $k \to 0$, one parameterizes $G_{aa}(q,\omega_n)$ by $\tilde{G}(q,\omega_n)$ and $G_{a\neq b}(q,\omega_n)$ par $G(q,u)$, $0 < u < 1$, and in a similar way $B_{a\neq b}(\tau) = \langle [u^a(x,\tau) - u^b(x,0)]^2)\rangle/N$ by $\tilde{B}(\tau)$ and $B(u)$ independently of $\tau$. Using the inversion rules of hierarchical matrices one obtains the saddle point equations (directly written in the limit $k \to 0$) by minimizing the variational free energy $F^{\rm var}$ (19),

$$G_c^{-1}(q,\omega_n) = \sum_b G_{ab}^{-1}(q,\omega_n) = cq^2 + M\omega_n^2 \quad (20)$$

$$+\frac{2}{\hbar}\int_0^{\beta\hbar} d\tau(1-\cos(\omega_n\tau))\left(\hat{V}'\left(\tilde{B}(\tau)\right) - \int_0^1 du\hat{V}'(B(u))\right)$$

$$\sigma(u,\omega_n) = \delta_{n,0}\frac{2\beta\hbar}{\hbar}\hat{V}'(B(u))$$

with

$$B(u) = \frac{2\hbar}{\beta\hbar}\int_q \sum_{n\neq 0} G_c(q,\omega_n) + \frac{2\hbar}{\beta\hbar}\int_q \left(\tilde{G}(q,\omega_n=0) - G(q,\omega_n=0,u)\right) \quad (21)$$

$$\tilde{B}(\tau) = \frac{2\hbar}{\beta\hbar}\sum_n \int_q G_c(q,\omega_n)(1-\cos(\omega_n\tau)) \quad (22)$$

with $\hat{V}(B) = -\rho_0^2\sum_K \Delta_K \exp(-BK^2/2)$. An interesting property of the above saddle point (21) is thus that replica symmetry breaking is confined [18] to the mode $\omega_n = 0$. The equation for this mode

$$\sigma(u,\omega_n) = \delta_{n,0}\frac{2\beta\hbar}{\hbar}\hat{V}'(B(u)) \quad (23)$$

has been studied previously [18] (notice that this is identical for the one for a model with point-like disorder in $d$ dimensions studied in [19,10]). For the potentials with power law correlators $\hat{V}(x) = gx^{1-\gamma}/(2(1-\gamma))$ there are two generic cases: long range correlations $\gamma(1-2/d) < 1$ for which one has full replica symmetry breaking (RSB) and short range correlations $\gamma(1-2/d) > 1$, for which one has a one step RSB (and a transition to high temperature replica symmetric (RS) phase). For the single cosine model defined by (15) where the sum over $K$ in $R(u)$ is restricted to the lowest harmonic, there is a one step RSB solution for $d \leq 2$ and a full RSB solution for $d > 2$. More realistic models involving several length scales include several of the above regimes [10]. The previous analysis of this equation (23) revealed the existence of a breakpoint $u_c$ such that $\sigma(u) = \sigma(u_c)$ for $u \geq u_c$. In the case of a full RSB solution, $\sigma(u)$ is a continuously varying function of $u$ for $u < u_c$. In $d \leq 2$, the single cosine model is instead described, in the low temperature phase, by a one step RSB solution such that $\sigma(u) = 0$ for $u < u_c$.

Independently of the RSB scheme, one can write the variational equations

$$G_c^{-1}(q,\omega_n) = cq^2 + M\omega_n^2 + \Sigma(1-\delta_{n,0}) + I(\omega_n) \quad (24)$$

$$I(\omega_n) = \frac{2}{\hbar}\int_0^{\beta\hbar} d\tau\,(1-\cos(\omega_n\tau))\left(\hat{V}'(\tilde{B}(\tau)) - \hat{V}'(B)\right)$$

$$B = B(u > u_c) = \frac{2\hbar}{\beta\hbar}\sum_n \int_q \frac{1}{cq^2 + M\omega_n^2 + \Sigma + I(\omega_n)}$$

together with the expression of $\tilde{B}(\tau)$ (22) and where we have used the definitions $\Sigma = [\sigma](u_c)$, $[\sigma](u) = u\sigma(u) - \int_0^u dv\sigma(v)$ together with the equation (23).

### 3.2 Solution of the variational equations

The general way to study the equation (23) has been presented in details in [18]. Here we present only the main results relevant for our study.

#### 3.2.1 Periodic structures in d > 2: marginally stable solution

In that case there exists a full RSB solution, the functions $B(u)$, $\sigma(u)$ and $[\sigma](u)$ being obtained by elimination from the system

$$1 = -4\hat{V}''(B(u))\mathcal{J}_2([\sigma](u)) \quad (25)$$

$$\sigma(u) = \frac{2\beta\hbar}{\hbar}\hat{V}'(B(u))$$

$$\mathcal{J}_n(x) = \int_q \frac{1}{(cq^2+x)^n}.$$

Once this system (26) is solved for $u < u_c$, the constants $\Sigma$ and $B$ are unambiguously determined by equation (24) together with the so-called marginality condition

$$1 = -4\hat{V}''(B)\mathcal{J}_2(\Sigma) \quad (26)$$

$$\Leftrightarrow \Sigma^{(4-d)/2} = \frac{\alpha_d}{c^{d/2}}\hat{V}''(B) \quad , \quad \alpha_d = \frac{\pi K_d(d-2)}{\sin\pi d/2}$$

where the second line in (26) is valid only in the infinite UV cut-off limit. In order to understand the finite temperature behavior of the variational equations below $u_c$, it is useful to write the equations (26) in terms of the rescaled variable $w = \beta\hbar u/\hbar$. One thus has

$$w = \frac{\beta\hbar u}{\hbar} \quad (27)$$

$$\sigma(u) = \frac{\beta\hbar}{\hbar}s(w) \quad (28)$$

$$[\sigma](u) = [s](w) \quad (29)$$

where the function $[s](w) = ws(w) - \int_0^w dvs(v)$ is $\hbar$ and $\beta\hbar$ *independent*. Indeed, differentiating (26), one obtains that it is implicitly defined through:

$$w = 4\frac{(\mathcal{J}_2([s]))^3}{\mathcal{J}_3([s])}\hat{V}'''\left(\left(-\hat{V}''\right)^{-1}\left(\frac{1}{\mathcal{J}_2([s])}\right)\right). \quad (30)$$

Notice that for the single cosine model corresponding to $\hat{V}(B) = -W\exp(-K^2B/2)$ one has $\hat{V}'''((-\hat{V}'')^{-1}(x)) = K^2x/2$, and equation (30) gives back $[s](w) = ((4-d)w/(2K^2c_d \times c^{-d/2}))^{2/(d-2)}$ with $c_d = (2-d)\pi^{1-\frac{d}{2}}/(2^{d+1}\sin(\pi d/2) \times \Gamma[d/2])$ (e.g. $c_3 = 1/(8\pi)$) [10]. The function $[\sigma](u) = [s](w)$ being independent of $\hbar$ and $\beta\hbar$ (30), it follows from (26) that $B(u) = \mathcal{B}(w)$ is also *independent* of $\hbar$ and $\beta\hbar$ below the breakpoint $u < u_c$. The equation (30) written at the breakpoint gives

$$w_c \equiv w_c(\Sigma) = \frac{\beta\hbar u_c}{\hbar} = 4\frac{(\mathcal{J}_2(\Sigma))^3}{\mathcal{J}_3(\Sigma)}\hat{V}'''(B). \qquad (31)$$

Finally, once the equation (30) is solved, $w_c$, $\Sigma$, $I(\omega_n)$ which all *depend* both on $\hbar$ and $\hbar$ are determined by equations (24), (26) and (31) together with the definition (22). Notice also that combining (26) and (31), one obtains

$$w_c\delta B + 2\mathcal{J}_2(\Sigma)\delta\Sigma = 0 \qquad (32)$$

where $\delta$ stands for an infinitesimal variation: a useful identity, only valid for a full RSB solution, in the following computations.

#### 3.2.2 Periodic structures in d ≤ 2: thermodynamics vs. marginality

The case $d \leq 2$ is more subtle, since as noticed previously [18] the saddle point equations admit two solutions. Which solution to choose was shown to be important for the transport properties and will of course be important for the specific heat as well as will be discussed in Section 4.2. Let us examine here the two possible solutions.

For the single cosine model corresponding to $\hat{V}(B) = -W\exp(-K^2B/2)$, in dimension $d \leq 2$, the variational equation (23) admits a one step RSB solution given by [10]

$$[\sigma](u) = 0, \quad u < u_c \qquad (33)$$

$$[\sigma](u) = \Sigma = \beta\hbar u_c\frac{2}{\hbar}\hat{V}'(B), \quad u \geq u_c. \qquad (34)$$

In this one step RSB case, one can use only (33, 24) to determine the three quantities $B, \Sigma$ and $u_c$: thus one equation is missing. In the statics, the breakpoint is then usually obtained by minimizing the variational free energy $F^{\rm var}$ with respect to $u_c$, the so called *thermodynamical* condition. It leads to in $d < 2$ [18]

$$\beta\hbar u_c^{\rm th} = K^2\hbar\frac{2-d}{d}\mathcal{J}_1\left(\Sigma^{\rm th}\right)$$
$$\Leftrightarrow w_c^{\rm th} = -K^2\frac{\alpha_d}{2d}\left(\Sigma^{\rm th}\right)^{(d-2)/2}c^{-d/2} \qquad (35)$$

where the second line in (35) is valid in the infinite UV cut-off limit with $\alpha_d$ given in (26). However, it is known that this condition (35) gives incorrectly the behavior of dynamical quantities such that the conductivity [18]. A distinct choice is to impose the marginality condition (26) which, given the relation (33)

$$1 = -4\hat{V}''(B^{\rm mg})\mathcal{J}_2(\Sigma^{\rm mg})$$
$$w_c^{\rm mg} = -K^2\frac{\alpha_d}{4}\Sigma^{(d-2)/2}c^{-d/2} \qquad (36)$$

which allows to obtain the correct dynamical behavior. One can show, using a Keldysh mean field approach and performing analytical continuation to imaginary time [22], that this is indeed the correct solution from the dynamical point of view, i.e., if one considers, in an infinite system, the large time limit where time translational invariance and equilibrium fluctuation dissipation theorem hold.

Finally, for the special case $d = 2$, these two conditions (35) and (36) coincide and there is no ambiguity in that case. Therefore, in the following we will treat this case together with the full RSB one in $d > 2$.

### 3.3 Internal energy ⟨H⟩ within variational method

In this section we use the variational method, and identities valid at the saddle point, to derive a compact and useful expression for the internal energy $\langle H\rangle$ (15) which is analyzed in the following Sections. The idea is indeed to compute the averaged values in (15) with the trial gaussian action $S_0$ (17) instead of the exact one $S^{\rm rep}$ (16). One can show that thanks to the variational equations, it is equivalent to compute $C_v(T)$ using the variational free energy (19) $C_v(T) = -T\partial^2F^{\rm var}/\partial T^2$ instead of the exact one.

We start by deriving some identities which will be useful below to replace the kinetic term $\propto \int_q \sum_n cq^2\tilde{G}(q,\omega_n)$ in (15) by a more convenient one. First we remind that $\tilde{G}(q,\omega_n) = G_c(q,\omega_n)$ for $\omega_n \neq 0$ (a consequence of the $\tau$-independence of the disorder (19)). Using simply the definition of $\sigma_{ab}(q,\omega_n)$ (17) and $G_c^{-1}(q,\omega_n)$ (21) one has

$$\sigma_{aa}(q,\omega_n) = cq^2 - G_c^{-1}(q,\omega_n) - \delta_{n,0}\sum_{b\neq a}\sigma_{ab}. \qquad (37)$$

Using (37) together with the identity $\sum_b G_{ab}(q,\omega_n)$ $G_{bc}^{-1}(q,\omega_n) = \delta_{ac}$, one obtains in the limit $k \to 0$

$$cq^2\tilde{G}(q,\omega_n) = cq^2G_c(q,\omega_n) \qquad (38)$$
$$+\delta_{n,0}\frac{2\beta\hbar}{\hbar}\int_0^1 du\left(\tilde{G}(q,\omega_n=0) - G(q,u)\right)\hat{V}'(B(u))$$

where we have used $G_c^{-1}(q,\omega_n = 0) = cq^2$ and the variational equation (23). Using the variational equation for $I(\omega_n)$ and the identity $\int_0^1 duV'B(u) = V'(B) - \frac{\hbar}{2\beta\hbar}\Sigma$ one has

$$\frac{\hbar}{2\beta\hbar}\sum_n G_c(q,\omega_n)(\Sigma(1-\delta_{n,0}) + I(\omega_n)) = \qquad (39)$$
$$\frac{1}{2\hbar}\int_0^{\beta\hbar}d\tau\tilde{B}(\tau)\hat{V}'\left(\tilde{B}(\tau)\right) - \sum_{n\neq 0}\int_0^1 duG_c(q,\omega_n)\hat{V}'(B(u)).$$

Combining (38, 39) and using the definition of $B(u)$ (24) one obtains

$$\frac{\hbar}{2\beta\hbar}\sum_n \int_q cq^2 \tilde{G}(q,\omega_n) \tag{40}$$
$$= \frac{\hbar}{2\beta\hbar}\sum_n \int_q \frac{cq^2+\Sigma+I(\omega_n)}{cq^2+\Sigma+M\omega_n^2+I(\omega_n)}$$
$$-\frac{1}{2\hbar}\int_0^{\beta\hbar} d\tau \tilde{B}(\tau)\hat{V}'(\tilde{B}(\tau)) + \frac{\beta\hbar}{2\hbar}\int_0^1 du B(u)\hat{V}'(B(u)).$$

This last expression (40) allows to write a compact expression for $\overline{\langle H\rangle}$ (15) computed within the variational method under the form

$$\frac{1}{\Omega}\overline{\langle H\rangle} = \frac{\hbar}{\beta\hbar}\sum_n \int_q \frac{cq^2+\Sigma+I(\omega_n)}{cq^2+\Sigma+M\omega_n^2+I(\omega_n)} \tag{41}$$
$$+\int_0^{\beta\hbar}\frac{d\tau}{\hbar}\left[F\left(\tilde{B}(\tau)\right)-F(B)\right]-\int_0^{w_c} dw[F(\mathcal{B}(w))-F(B)]$$

where $F(X) = \hat{V}(X) - \frac{X}{2}\hat{V}'(X)$ and $w_c$ given in (31). Although this form is compact and convenient, its temperature dependence is hard to extract, and we will resort to an expansion in powers of $\hbar$.

## 4 Semi-classical expansion: lowest order

In this section we extract the temperature dependence of the specific heat from (41). In order to do so we use an interesting property of the variational equations (24, 26): the solution, as well as the internal energy (41) can be organized in an expansion in $\hbar$ keeping $\beta\hbar$ fixed. To tackle analytically these variational equations, we will thus organize our calculations using this expansion for any quantity (not designated by a calligraphic letter) $Q(\hbar,\beta\hbar) = \sum_0^\infty \hbar^n Q_n(\beta\hbar)$.

The calculation of the specific heat to a given order in the expansion requires the knowledge of the saddle point solution quantities, including $I(\omega_n)$, to the same order. Such an analysis of the saddle point solution was performed previously [18] only to lowest order. We start by recalling this analysis and extracting from it the specific heat to lowest order.

In order to check whether the obtained temperature dependence is correctly captured by the lowest order calculation, we examine in Section 5 the higher orders.

### 4.1 d $\geq$ 2

The analysis of the variational equations (24, 26) leads to the following equations for $I_0(\omega_n)$

$$I_0(\omega_n) = \tag{42}$$
$$-4\hat{V}''(0)\left(\mathcal{J}_1(\Sigma_0) - \mathcal{J}_1\left(\Sigma_0+\frac{c}{v^2}\omega_n^2+I_0(\omega_n)\right)\right)$$

and for $\Sigma_0$

$$1 = -4\hat{V}''(0)\mathcal{J}_2(\Sigma_0). \tag{43}$$

Because of the marginality condition (43), the function $I_0(\omega_n)$ is non analytic, its low frequency behavior being given by

$$I_0(\omega_n) \sim \sqrt{\frac{c}{v^2}\frac{\mathcal{J}_2(\Sigma_0)}{\mathcal{J}_3(\Sigma_0)}}|\omega_n| + \mathcal{O}\left(\omega_n^2\right). \tag{44}$$

The marginality condition (43) thus leads to a gapless excitation spectrum. Indeed, the low frequency behavior of the analytical continuation $I_0(\omega_n \to -i\omega + 0^+) = I_0'(\omega) + iI_0''(\omega)$ reads

$$I_0'(\omega) \sim \mathcal{A}\omega^2 \quad , \quad I_0''(\omega) \sim -\mathcal{B}\omega \tag{45}$$

with $\mathcal{A} = \frac{c}{v^2}(1-\frac{\mathcal{J}_2\mathcal{J}_4}{2\mathcal{J}_3^2})$ and $\mathcal{B} = \sqrt{\frac{c}{v^2}\frac{\mathcal{J}_2}{\mathcal{J}_3}}$, where $\mathcal{J}_n = \mathcal{J}_n(\Sigma_0)$. Notice also that at this lowest order the equations (42, 43) show that $I_0(\omega_n)$ and $\Sigma_0$ are independent of $\beta\hbar$ ($I_0(\omega_n)$) depends of course *implicitly* of $\beta\hbar$ through $\omega_n$). One then obtains the lowest order expansion from (41) $\overline{\langle H\rangle}/\Omega = H_0 + \hbar H_1(\beta\hbar) + \mathcal{O}(\hbar^2)$

$$H_0 = -2F'(0)\mathcal{J}_1(\Sigma_0) - \int_0^{w_{c0}} dw[F(\mathcal{B}(w)) - F(0)]$$
$$H_1 = \frac{1}{\beta\hbar}\sum_n \int_q \frac{cq^2+\Sigma_0+I_0(\omega_n)}{cq^2+\frac{c}{v^2}\omega_n^2+\Sigma_0+I_0(\omega_n)}$$
$$+F'(0)\left[2\Sigma_1\mathcal{J}_2\left(\Sigma_0\right)+w_{c0}B_1\right] \tag{46}$$

where we have used that $\mathcal{B}(w)$ is $\hbar$-independent for $w < w_c$ as well as $\lim_{w\to w_c-}\mathcal{B}(w) = B$ in the case of a full RSB solution and $F''(0) = 0$. From (32), the last terms in expression (46) just cancel leading simply to

$$H_1 = \frac{1}{\beta\hbar}\sum_n \int_q \frac{cq^2+\Sigma_0+I_0(\omega_n)}{cq^2+\frac{c}{v^2}\omega_n^2+\Sigma_0+I_0(\omega_n)}. \tag{47}$$

From equation (31), one has that $w_{c0} \equiv w_{c0}(\Sigma_0)$ is independent of $\beta\hbar$, as well as $\mathcal{B}(w)$ for $w < w_c$ such that $H_0$ is $\beta\hbar$ independent. To compute the specific heat to lowest order in this $\hbar$ expansion, $C_v = C_{v0}(\beta\hbar) + \mathcal{O}(\hbar)$, one thus focuses on $H_1$, whose temperature dependence is contained in the Matsubara frequencies. Transforming the discrete sum over Mastsubara frequencies in (47) in an integral one obtains,

$$\hbar H_1 = \int_{-\infty}^{+\infty}\frac{d\omega}{\pi}\hbar\omega\rho(\omega)f_B(\omega) \tag{48}$$
$$\rho(\omega) = \frac{c}{v^2}\omega\int_q \mathrm{Im}G_c(q,\omega_n\to -i\omega+0^+)$$
$$= \int_q \frac{c}{v^2}\omega\frac{-I_0''(\omega)}{\left(cq^2-\frac{c}{v^2}\omega^2+\Sigma_0+I_0'(\omega)^2+(I_0''(\omega))^2\right)} \tag{49}$$

where $\rho(\omega)$ is the density of states. All the temperature dependence in (48) is now contained in the Bose factor. It

follows from (48) that

$$C_{v0}(T) = (\beta\hbar)^2 \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \frac{\rho(\omega)\omega^2}{\sinh^2 \beta\hbar\omega/2}. \quad (50)$$

The low temperature behavior of the integral in (50) is governed by the low frequency behavior of the density of states $\rho(\omega) \sim \omega^2$ (49, 45), which then leads to $C_{v0}(T) \sim T^3$ in all dimensions $d \geq 2$. To this lowest order, there is thus no linear nor quadratic term in $T$ in the specific heat. The specific heat has the dimension of the inverse of a volume, its low temperature behavior is then in general characterized by a typical length scale and an energy scale. In our problem, a natural energy scale $T_p$ is given by the pinning frequency $\omega_p$ with $T_p = \hbar\omega_p = \hbar v\sqrt{\Sigma_0/c} = \hbar v/R_c$, where $R_c$ is the Larkin length. From (50), one obtains

$$C_v(T) \sim \frac{4\pi^4}{15} K_d R_c^{-d} F_{C_v}[R_c/a] \left(\frac{T}{T_p}\right)^3 + \mathcal{O}\left(\hbar, (T/\hbar)^4\right) \quad (51)$$

where $F_{C_v}(x)$ is a scaling function with asymptotic behaviors given by

$$F_{C_v}[x] \sim \frac{1}{\sqrt{4-d}} \left|\frac{d-2}{\sin \pi d/2}\right| \quad x \gg 1 \quad (52)$$

$$F_{C_v}[x] \sim \frac{2^{d+1}}{d} \pi^{d-1} x^d \quad x \ll 1. \quad (53)$$

Thus, at very weak disorder, $R_c \gg a$ (52), and the typical volume associated to the specific heat is $R_c^d$, although at stronger disorder $R_c \ll a$ (53) it becomes $a^d$, the scaling function $F_{C_v}(x)$ describing the crossover from one regime to the other. Notice also that, in $d = 3$, in the limiting case $R_c/a \to \infty$, (51) together with (52) and $K_3 = 1/(2\pi^2)$ give back exactly the Debye formula (9) for pure phonons.

Our results are thus at variance with the naive linear $T$ expected from the two level system phenomenology. Contrarily to the pure case, the specific heat is now proportional to $T^3$ independently of the dimension. This specific heat reflects the density of states of the modes of vibration of the disordered system. A formula such as (50) shows that the specific heat would be the same than the one for free independent vibrational modes with a density of states $\rho(\omega) \sim \omega^2$. Note however, that since here the modes are not independent the precise calculation of the specific heat cannot be done naively and requires the full calculation of the energy, as we have performed in the present paper.

### 4.2 Periodic structure in d < 2

In that case, where there is a one-step RSB solution, the equation for $I_0(\omega_n)$ is also given by (42) but the choice of the thermodynamical condition (35) leads to [18] $I_0^{\rm th}(\omega_n) \propto \omega_n^2$, for small $\omega_n$, and indicates a gap in the energy spectrum.

Instead, if one uses the marginality condition (36), the behavior is then similar to the previous full RSB solution with [18] $I_0^{\rm mg}(\omega_n) \propto |\omega_n|$, and describes a gapless energy spectrum. As discussed in the previous Section 3.2.2, the marginality condition is the consistent prescription (within this Mean Field approach) to compute dynamical quantities.

We now turn to the computation of the internal energy in both cases (35, 36). In that case, where there is a one step RSB solution, the function $\mathcal{B}(w)$ is now discontinuous at the breakpoint $w_c$. From this discontinuity results an other contribution to $H_1$ compared to expression for the full RSB case (46). Indeed in that case one has in both cases:

$$\begin{aligned} H_0 &= -2F'(0)\mathcal{J}_1(\Sigma_0) + w_{c0}F(0) \\ H_1 &= \frac{1}{\beta\hbar}\sum_n \int_q \frac{cq^2 + \Sigma_0 + I_0(\omega_n)}{cq^2 + \frac{c}{v^2}\omega_n^2 + \Sigma_0 + I_0(\omega_n)} \\ &+ F'(0)[2\Sigma_1\mathcal{J}_2(\Sigma_0) + w_{c0}B_1] + w_{c1}F(0) \end{aligned} \quad (54)$$

where $H_0$ is still temperature independent. Notice that here, one can not use the general formula (32) valid only for a full RSB. If one uses the thermodynamical condition, one has for the periodic case $\hat{V}(B) = -W\exp(-K^2B/2)$

$$\begin{aligned} {H_1}^{\rm th} &= \frac{1}{\beta\hbar}\sum_n \int_q \frac{cq^2 + \Sigma_0^{\rm th} + I_0^{\rm th}(\omega_n)}{cq^2 + \frac{c}{v^2}\omega_n^2 + \Sigma_0^{\rm th} + I_0^{\rm th}(\omega_n)} + \Delta{H_1}^{\rm th} \\ \Delta{H_1}^{\rm th} &= \hat{V}(0)\left(\frac{{w_c^{\rm th}}_1}{2} + \frac{\Sigma_1}{2\Sigma_0}{w_c^{\rm th}}_0 - \frac{K^2}{2}\Sigma_1^{\rm th}\mathcal{J}_2\left(\Sigma_0^{\rm th}\right)\right) \\ &= \hat{V}(0)\left(\frac{d}{4}{w_c^{\rm th}}_0\frac{\Sigma_1^{\rm th}}{\Sigma_0^{\rm th}} - K^2\frac{\Sigma_1^{\rm th}}{\Sigma_0^{\rm th}}K_d\frac{2-d}{8}\frac{\pi\left(\Sigma_0^{\rm th}\right)^{(d-2)/2}}{\sin\pi d/2}\right) \\ &= 0 \end{aligned} \quad (55)$$

where we have used (35). One thus obtains formally the same expression as previously obtained for the full RSB case (47). But here, as $I_0^{\rm th}(\omega_n) \propto \omega_n^2$, describing a gapped excitation spectrum, one obtains that the specific heat ${C_v^{\rm th}}_0(T)$ vanishes *exponentially* at low $T$. If one imposes instead the marginality condition equation (36) one obtains, using (33)

$$\begin{aligned} H_1^{\rm mg} &= \frac{1}{\beta\hbar}\sum_n \int_q \frac{cq^2 + \Sigma_0^{\rm mg} + I_0^{\rm mg}(\omega_n)}{cq^2 + \frac{c}{v^2}\omega_n^2 + \Sigma_0^{\rm mg} + I_0^{\rm mg}(\omega_n)} + \Delta{H_1}^{\rm mg} \\ \Delta{H_1}^{\rm mg} &= \frac{\Sigma_0^{\rm mg}B_1^{\rm mg}}{2}\left(\frac{1}{2} - \frac{\hat{V}(0)\hat{V}''(0)}{\hat{V}''(0)^2}\right. \\ &\left. + \frac{1}{4-d}\left(\frac{2\hat{V}(0)\hat{V}'''(0)}{\hat{V}'(0)\hat{V}''(0)} - \frac{\hat{V}'(0)\hat{V}'''(0)}{\hat{V}''(0)^2}\right)\right) \\ &= \Sigma_0^{\rm mg}\frac{d-2}{2(4-d)}\frac{\hbar}{\beta\hbar}\sum_n \int_q \frac{1}{cq^2 + \frac{c}{v^2}\omega_n^2 + \Sigma_0^{\rm mg} + I_0^{\rm mg}(\omega_n)}. \end{aligned} \quad (56)$$

This results in the low temperature behavior of ${C_v^{\rm mg}}_0(T)$

$$C_v^{\rm mg}{}_0(T) = (\beta\hbar)^2 \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \frac{\rho^{\rm mg}(\omega)\left(\omega^2 + (\Sigma_0^{\rm mg})\frac{d-2}{2(4-d)}\right)}{\sinh^2 \beta\hbar\omega/2}$$

$$\sim \Sigma_0^{\rm mg} \frac{d-2}{2(4-d)} \frac{T}{\hbar} + O\left(\left(\frac{T}{\hbar}\right)^3\right) \quad (57)$$

which is linear in $T$ but *negative* for $d < 2$.

This shows explicitly in that case the inconsistency of the marginality condition (26) to compute thermodynamical quantities. That for such glassy system one has carefully to distinguish between thermodynamics and dynamics quantities occurs also in other physical quantities. The capacitance (or compressibility) is also different depending on whether one considers the thermodynamics one or the small frequency one [13,12,23]. The calculation of the specific heat clearly shows that for a thermodynamic quantity one should not use the criterion given by the marginality condition. The discussion of Section 3.2.2 suggests that the specific heat defined by the marginality condition (i.e. negative in the present case) could correspond to an experiment on a aging system using slow time dependent heat.

Using the thermodynamic saddle point equation, on the other hand gives in one dimension (more generally for $d < 2$) an exponentially small specific heat at low temperature due to a gap. This is clearly an artefact of the variational approach. Corrections away from mean-field will most likely transform this gap into a pseudo gap, and yield a $T^\alpha$ behavior for the specific heat. It is reasonable to surmise that the exponent $\alpha$ is larger than three and thus that the variational method attempts to reproduce such a large power by artificially inducing a gap. Indeed some studies of one-dimensional systems [24–26] suggest a powerlaw behavior $\rho(\omega) \sim \omega^4$. Such a density of state would lead to a $T^5$ behavior for the specific heat.

## 5 Beyond the leading order

Given the fact that the linear temperature dependence disappears in (50), it is important to know whether this property holds to higher orders. We thus derive explicitly the solution to the next order. In doing so we unravel a structure of the variational solution not present to leading order.

### 5.1 A need for a singular self energy in d $\geq$ 2

We first need to extend the study of the variational equations to next order, allowing to extract $\Sigma_1$ and $I_1(\omega_n)$. To this order, as we will see, this quantity becomes $\beta\hbar$ dependent. From (24) and (26), one obtains

$$\Sigma_1 = \frac{\hat{V}'''(0)}{-8\left(\hat{V}''(0)\right)^2 \mathcal{J}_3(\Sigma_0)} B_1, \quad B_1 = \frac{2}{\beta\hbar}\sum_n \mathcal{K}_1(\omega_n)$$

$$\Sigma_1 + I_1(\omega_n) = \frac{1}{1+4\hat{V}''(0)\mathcal{K}_2(\omega_n)}\left(B_1 I_0(\omega_n)\frac{\hat{V}'''(0)}{\hat{V}''(0)}\right.$$

$$\left. + \frac{4\hat{V}'''(0)}{\beta\hbar}\sum_m \mathcal{K}_1(\omega_m)(\mathcal{K}_1(\omega_m) - \mathcal{K}_1(\omega_m+\omega_n))\right) \quad (58)$$

$$\mathcal{K}_p(\omega_n) = \int_q \frac{1}{(cq^2 + M\omega_n^2 + \Sigma_0 + I_0(\omega_n))^p}. \quad (59)$$

Note that in the limit $\omega_n \to 0$ the denominator in the l.h.s of (58) behaves as $-8\hat{V}''(0)I_0(\omega_n)\mathcal{J}_3(\Sigma_0)$ and thus the first term yields exactly $\Sigma_1$ in this limit. Given the complexity of the second term in the l.h.s of (58), we analyze it by expanding it at low temperature, i.e. high $\beta$. This expansion can be performed using the Euler-MacLaurin formula or, equivalently, a spectral representation of the Green function $G_c(q,\omega_n)$ (see Appendix 6 for details). It yields

$$\frac{4\hat{V}'''(0)}{(\beta\hbar)}\sum_m \mathcal{K}_1(\omega_m)(\mathcal{K}_1(\omega_m) - \mathcal{K}_1(\omega_n+\omega_n)) \quad (60)$$

$$= 4\hat{V}'''(0)\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\mathcal{K}_1(\omega)(\mathcal{K}_1(\omega) - \mathcal{K}_1(\omega+\omega_n))$$

$$- \left(\frac{T}{\hbar}\right)^2 I_0(\omega_n)\frac{\hat{V}'''(0)}{\hat{V}''(0)}\frac{2\pi}{3}\int_q A_0'(q,0) + O\left(\left(\frac{T}{\hbar}\right)^4\right)$$

with the definition

$$A_0'(q,\omega) = \partial_\omega A_0(q,\omega)$$
$$A_0(q,\omega) = \mathrm{Im}G_c\left(q,\omega_n \to -i\omega + 0^+\right) \quad (61)$$

and where we have used the self-consistent equation for $I_0(\omega_n)$ (42). Although the first term in (60), corresponding to the dominant one in the limit $(\beta\hbar) \to \infty$, behaves like $\omega_n^2$ and leads to a linear term, $\propto|\omega_n|$ in $I_1(\omega_n)$ (notice that $I_0(\omega+\omega_n)$ is well defined and can be explicitly computed (42)), the second term in (60) is *linear* in $\omega_n$, and produces a new term $\tilde{C}$ in $I_1(\omega_n)$ (we remind that $I_1(\omega_n = 0) = 0$ by definition (24)):

$$I_1(\omega_n) = \tilde{C}(1-\delta_{n,0}) + \tilde{I}_1(\omega_n) \quad (62)$$

$$\tilde{C} = \left(\frac{T}{\hbar}\right)^2 \frac{\pi}{12}\frac{\hat{V}'''(0)}{\mathcal{J}_3(\hat{V}''(0))^2}\int_q A_0'(q,0) + \mathcal{O}\left(\left(\frac{T}{\hbar}\right)^4\right)$$

where $\tilde{I}_1(\omega_n)$ is a well defined function such that $\tilde{I}_1(\omega_n) \propto |\omega_n|$ for small $\omega_n$. Notice also that this term $\tilde{C}$ arises only at *finite temperature*. Finally, combining equation (60) and

$$B_1 = \frac{2}{\beta\hbar}\sum_n \mathcal{K}_1(\omega_n) = 2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathcal{K}_1(\omega) \quad (63)$$

$$+ \left(\frac{T}{\hbar}\right)^2 \frac{2\pi}{3}\int_q A_0'(q,0) + O\left(\left(\frac{T}{\hbar}\right)^4\right)$$

it follows that the terms of order $(T/\hbar)^2$ in (58), for $\omega_n \neq 0$, exactly cancel leading to

$$\Sigma_1 + I_1(\omega_n) = \frac{2\hat{V}'''(0)}{1+4\hat{V}''(0)\mathcal{K}_2(\omega_n)} \times \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\mathcal{K}_1(\omega)\left(\frac{I_0(\omega_n)}{\hat{V}''(0)} + 2(\mathcal{K}_1(\omega)-\mathcal{K}_1(\omega+\omega_n))\right) + O\left(\left(\frac{T}{\hbar}\right)^4\right), \quad \omega_n \neq 0. \tag{64}$$

Note that, although at this order $\Sigma_1$ and $I_1(\omega_n)$ admits a low temperature expansion involving only even powers of $\beta\hbar$ (see Appendix 6 for more details), the presence of the peculiar term $\tilde{C}$ in (62) generates however odd powers of $\beta\hbar$ at higher orders of this semi-classical expansion. Indeed considering the definition of $B$ (24), one has to handle with care the sum over Matsubara frequencies and isolate the mode $\omega_n = 0$ in the following way

$$B = \frac{\hbar^2}{\beta\hbar}\tilde{C}\int_q \frac{1}{(cq^2+\Sigma_0+\Sigma_1)\left(cq^2+\Sigma_0+\Sigma_1+\tilde{C}\right)} + \frac{\hbar}{\beta\hbar}\sum_n \int_q \frac{1}{cq^2+\Sigma_0+\hbar\left(\Sigma_1+\tilde{C}\right)+I_0(\omega_n)+\hbar\tilde{I}_1(\omega_n)}. \tag{65}$$

Under this form, the sum over $\omega_n$ can be performed safely, and will itself admit a low $T$ expansion in even powers of $\beta\hbar$. However, the first term in (65) will generate a contribution to $\Sigma_2$ (and consequently also to $I_2(\omega_n)$) proportional to $(T/\hbar)^3$, thus generating odd powers of $T$.

As we will discuss in the next subsection, (64) ensures that the temperature dependence given by the lowest order term is indeed correct. The property (64) can be shown to hold in fact to all orders [9] in $\hbar$ and to apply also to other models of glasses.

### 5.2 Specific heat

We first focus on the expansion of $\langle H\rangle$ (41) to order $\hbar^2$ and compute $H_2$. The first term in (41) is simply expanded by substituting $\Sigma$ and $I(\omega_n)$ by their own expansion. We will leave it under this form (i.e. as in (47) by substituting $\Sigma_0 \to \Sigma_0+\hbar\Sigma_1$ and similarly $I_0(\omega_n) \to I_0(\omega_n)+\hbar I_1(\omega_n)$), which can then be easily expanded at low temperature. At this order $\mathcal{O}(\hbar^2)$ however, the second line of (41) leads to a non trivial contribution which, after some manipulations, can be written as

$$\langle H\rangle - H_0 = \tag{66}$$

$$\frac{\hbar}{\beta\hbar}\sum_n\int_q \frac{cq^2+\Sigma_0+\hbar\Sigma_1+I_0(\omega_n)+\hbar I_1(\omega_n)}{cq^2+M\omega_n^2+\Sigma_0+\hbar\Sigma_1+I_0(\omega_n)+\hbar I_1(\omega_n)}$$

$$-\hbar^2\frac{\hat{V}'''(0)}{\hat{V}''(0)}\frac{1}{2(\beta\hbar)^2}\sum_n \mathcal{K}_1(\omega_n)\sum_n I_0(\omega_n)\mathcal{K}_1(\omega_n) \tag{67}$$

$$+\frac{2\hbar^2}{3}\hat{V}'''(0)\frac{1}{(\beta\hbar)^2}\sum_{m,n}\mathcal{K}_1(\omega_n)\mathcal{K}_1(\omega_m)\mathcal{K}_1(\omega_n+\omega_n)$$

where $H_0$, which is $\beta\hbar$ independent is given in (46). From this expression of $\langle H\rangle$ to order $\mathcal{O}(\hbar^2)$ (66), we now want to compute the low temperature expansion to obtain the specific heat $C_v(T)$ to order $\mathcal{O}(\hbar)$. The analysis of the first term in (66) can be done similarly to $H_1$ (48). And as we showed previously, the finite temperature corrections to $\Sigma_1+I_1(\omega_n)$ behave as $T^4$ (64) (note that the peculiarity of the mode $\omega_n = 0$ disappears in the first term of (66)), one obtains that

$$\frac{\hbar}{\beta\hbar}\sum_n\int_q \frac{cq^2+\Sigma_0+\hbar\Sigma_1+I_0(\omega_n)+\hbar I_1(\omega_n)}{cq^2+M\omega_n^2+\Sigma_0+\hbar\Sigma_1+I_0(\omega_n)+\hbar I_1(\omega_n)} \propto \left(\frac{T}{\hbar}\right)^4 + O\left(\left(\frac{T}{\hbar}\right)^6\hbar^3\right). \tag{68}$$

Given the complexity of the calculations, we did not intend to compute the amplitude of this term. We now turn to the analysis of the low temperature behavior of the last two terms in (66). As the specific heat vanishes at zero temperature, the first non vanishing finite temperature correction in (66) is a priori of order $1/(\beta\hbar)^2$. As shown in Appendix A.3, it turns out that this term exactly cancels between these last two terms in (66), leading also to

$$-\hbar^2\frac{\hat{V}'''(0)}{\hat{V}''(0)}\frac{1}{2(\beta\hbar)^2}\sum_n\mathcal{K}_1(\omega_n)\sum_n I_0(\omega_n)\mathcal{K}_1(\omega_n) + \frac{2\hbar^2}{3}\hat{V}'''(0)\frac{1}{(\beta\hbar)^2}\sum_{m,n}\mathcal{K}_1(\omega_n)\mathcal{K}_1(\omega_m)\mathcal{K}_1(\omega_n+\omega_n) \propto \hbar^2\left(\frac{T}{\hbar}\right)^4. \tag{69}$$

Therefore, combining (68) and (69) together with (66), one obtains that

$$\langle H\rangle - H_0 \propto \left(\frac{T}{\hbar}\right)^4 + O\left(\left(\frac{T}{\hbar}\right)^6, \hbar^3\right) \tag{70}$$

$$C_v(T) \propto \left(\frac{T}{\hbar}\right)^3 + O\left(\left(\frac{T}{\hbar}\right)^5, \hbar^2\right) \tag{71}$$

which shows explicitly the cancellation of the term linear in $T$ in $C_v(T)$ up to order $\mathcal{O}(\hbar^2)$. The term (70) gives a correction to the amplitude of the $T^3$ term in the specific heat (50).

## 6 Conclusion

In this paper we have studied the specific heat of a disordered elastic system. Using a variational approach, we have shown that the leading temperature dependence of the specific heat is $C_v \propto T^3$ at low temperatures for dimensions $d \geq 2$. We have computed the prefactor of the $T^3$ law up to the second order in a semi-classical expansion in $\hbar$. It exhibits a dependence in the Larkin pinning length due to disorder. We have showed up to order $\hbar^2$

that linear temperature dependence cancels in the specific heat. This property is quite general and is proved to all orders using a different approach in reference [9] where similar consequences are shown to hold for quantum spin glass models.

We thus find results at variance with the commonly believed linear temperature dependence stemming from e.g. a two level system. In the context of structural glasses, the linear in $T$ term is also commonly inferred from anharmonicity and disordered elastic constants. In the present problem, these are highly irrelevant perturbations to the model considered here (1) and we thus do not expect that they will modify the low $T$ behavior we have obtained. They will however produce a linear in $T$ contribution to $C_v(T)$ in the *classical* regime [27].

The $T^3$ dependence of the specific heat also holds for non periodic manifolds, such as polymers or interfaces, whenever they can be solved by a continuous replica symmetry broken ansatz (or its limiting case of one step marginal RSB). Indeed the results we obtained in this paper do not rely on the periodicity of the elastic system. In the case of manifolds, the dependence of the specific heat we obtained becomes exact in the limit of an infinite number of components.

For periodic systems, the variational method is only an approximation, and it would thus be interesting to check whether the powerlaw $T^3$ predicted here, has corrections away from mean-field. In particular, the question whether soliton-like excitations, which may not be treated accurately by the present method, could reestablish a linear in $T$ contribution to $C_v(T)$ from two-level system argument remains an open question. It is however important to note that periodic systems have been shown to be stable, for disorder weaker than a finite threshold, to the proliferation of topological defects induced by disorder in dimensions $d > 2$. For point like disorder this leads to the existence of a Bragg glass phase [10,16,17]. In such a phase, one would expect the Gaussian approximation to be an accurate one, and thus the present results to hold. On the other hand since for $d \leq 2$ topological defects (soliton like excitations) are generated by disorder [10,28,29], they should be taken into account for the specific heat, leading most certainly to additional contributions compared to the ones of the purely Gaussian approximation. Such a check of soliton contributions would most likely need numerical investigations of this problem.

For $d < 2$ we showed that within the framework of the variational method it is incorrect to use the marginality condition criterion to compute the thermodynamic specific heat. The thermodynamic saddle point leads to an exponentially small specific heat for $d < 2$. As we discussed in this paper, this is most likely an artefact of the mean-field approximation. Corrections to mean-field should transform this gap into a pseudo-gap, with powerlaw density of states $\rho(\omega) \propto \omega^{\alpha-1}$ leading to a specific heat $T^\alpha$. The exponent $\alpha$ is likely to be larger than three in $d = 1$. The presence of a gap in the mean-field solution results from the best attempt of the variational method to mimic this high power. Such behavior in the density of states is indeed compatible with studies in one dimension [24–26], suggesting $\rho(\omega) \propto \omega^4$.

An important question is thus how this density of states evolves when going to higher dimensions. The calculations performed in the present paper suggest that for $d \geq 2$ the density of states becomes $\rho(\omega) \propto \omega^2$. Note that this is at variance with the recent proposal [30] that it remains $\rho(\omega) \propto \omega^4$ as in one dimension. If this is indeed correct it would raise the puzzling question to understand which mechanism can *lower* the density of modes compared to mean-field.

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## Appendix A: Evaluation of Matsubara sums at low temperature

In this section, we show in detail how to extract the low $T$ behavior of the sum over Matsubara frequencies in the equation for $\Sigma_1 + I_1(\omega_n)$ (58), which turns out to be crucial to understand the full structure of the saddle point solution. Although the standard way of studying such a sum is to use the spectral representation of the Green function, we show how to analyse it using less sophisticated method, namely the Euler Mac-Laurin formula. For this particular case, this method turns out to be very simple. Its equivalence with the conventional method is established in the last paragraph of this section.

### A.1 A first stage with Euler MacLaurin

The aim is to evaluate the low $T$ expansion of:

$$S = 2\pi \frac{T}{\hbar} \sum_n g(|\omega_n|) \tag{72}$$

with $\omega_n = 2\pi n T/\hbar$, a Matsubara frequency, and let us add a cutoff $|n| < N$, e.g. $N = \hbar\Omega/(2\pi T)$. We use the Euler-MacLaurin formula for any smooth enough function $f(x)$:

$$f(0) + 2\sum_1^{N-1} f(n) = 2\int_0^N dk f(k) - f(N) \tag{73}$$
$$+\frac{1}{6}\left(f'(N) - f'(0)\right) - \frac{1}{360}\left(f'''(N) - f'''(0)\right) + ...$$

Applying (73) to $f(k) = 2\pi(T/\hbar)g(2k\pi T/\hbar)$ gives:

$$S - S_c = \tag{74}$$
$$2\int_0^\Omega dx f(x) - \frac{1}{6}\left(\frac{2\pi T}{\hbar}\right)^2 g'(0) + \frac{1}{360}\left(\frac{2\pi T}{\hbar}\right)^4 g''(0)$$
$$S_c = -2\pi\frac{T}{\hbar} g(\Omega) + \frac{1}{6}\left(\frac{2\pi T}{\hbar}\right)^2 g'(\Omega) + ...$$

and if the above integral converges we can take $\Omega \to \infty$ and the boundary term $S_c$ vanishes.

### A.2 More sophisticated EML formula

We now want to evaluate a more complicated sum such as in (58):

$$S = 2\pi\frac{T}{\hbar}\sum_n g_1(|2\pi(n-m)T/\hbar|)g_2(|2\pi nT/\hbar|) \quad (75)$$
$$2\pi\frac{T}{\hbar}\sum_n g_1(|2\pi nT/\hbar|)g_2(|2\pi(n+m)T/\hbar|)$$

with $y = 2\pi mT/\hbar$, and where we leave aside the cutoff from the beginning, i.e. $n$ goes from $-\infty$ to $\infty$. We choose $m > 0$ but similar calculations hold also for $m < 0$. We use the decomposition:

$$S = S_1 + S_2 + S_3 \quad (76)$$
$$+2\pi\frac{T}{\hbar}g_1\left(|2\pi mT/\hbar|\right)g_2(0) + 2\pi\frac{T}{\hbar}g_1(0)g_2\left(|2\pi mT/\hbar|\right)$$

with

$$S_1 = 2\pi\frac{T}{\hbar}\sum_{n=-\infty}^{-1} g_1\left(|2\pi(n-m)T/\hbar|\right)g_2\left(|2\pi nT/\hbar|\right)$$
$$S_2 = 2\pi\frac{T}{\hbar}\sum_{n=1}^{m-1} g_1\left(|2\pi(n-m)T/\hbar|\right)g_2\left(|2\pi nT/\hbar|\right)$$
$$S_3 = 2\pi\frac{T}{\hbar}\sum_{n=m+1}^{\infty} g_1\left(|2\pi(n-m)T/\hbar|\right)g_2\left(|2\pi nT/\hbar|\right).$$

Applying the standard Euler-MacLaurin formula (73) to $S_1$ one obtains up to terms of order $\mathcal{O}((T/\hbar^4))$

$$S_1 = 2\pi\frac{T}{\hbar}\sum_{n=1}^{\infty} g_1\left(2\pi(m+n)T/\hbar\right)g_2\left(2\pi nT/\hbar\right)$$
$$= \int_0^\infty dx g_1(x+y)g_2(x) - \frac{1}{2}2\pi\frac{T}{\hbar}g_1(y)g_2(0)$$
$$+\left(2\pi\frac{T}{\hbar}\right)^2\frac{1}{12}\left(-g_1'(y)g_2(0) - g_1(y)g_2'(0)\right) \quad (77)$$

similarly to $S_2$

$$S_2 = 2\pi\frac{T}{\hbar}\sum_{n=1}^{m-1} g_1\left(2\pi(m-n)T/\hbar\right)g_2\left(2\pi nT/\hbar\right)$$
$$= \int_0^y dx g_1(y-x)g_2(x) - \frac{1}{2}2\pi\frac{T}{\hbar}g_1(y)g_2(0)$$
$$-\frac{1}{2}2\pi\frac{T}{\hbar}g_1(0)g_2(y)$$
$$+\left(2\pi\frac{T}{\hbar}\right)^2\frac{1}{12}\left(g_1'(y)g_2(0) - g_1(y)g_2'(0) - g_1'(0)g_2(y)\right.$$
$$\left.+g_1(0)g_2'(y)\right) \quad (78)$$

and $S_3$

$$S_3 = 2\pi\frac{T}{\hbar}\sum_{n=m+1}^{\infty} g_1\left(2\pi(n-m)T/\hbar\right)g_2\left(2\pi nT/\hbar\right)$$
$$= \int_y^\infty dx g_1(x-y)g_2(x) - \frac{1}{2}2\pi\frac{T}{\hbar}g_1(0)g_2(y)$$
$$+\left(2\pi\frac{T}{\hbar}\right)^2\frac{1}{12}\left(-g_1'(0)g_2(y) - g_1(0)g_2'(y)\right). \quad (79)$$

Collecting the terms in (77, 78, 79) yields finally

$$S = \int_0^\infty dx g_1(|x-y|)g_2(x) + \int_0^\infty dx g_1(x+y)g_2(x)$$
$$-2\left(2\pi\frac{T}{\hbar}\right)^2\frac{1}{12}\left(g_1(y)g_2'(0) + g_1'(0)g_2(y)\right) + \mathcal{O}\left((T/\hbar)^4\right). \quad (80)$$

### A.3 Euler Mc-Laurin vs. spectral representation

We want to compute the high $\beta$ expansion of the following Matsubara sum which enters the equation (58):

$$\tilde{S}(\omega_n) = \frac{1}{\beta\hbar}\sum_m \mathcal{K}_1(\omega_m)\mathcal{K}_1\left(\omega_n + \omega_m\right). \quad (81)$$

The Euler-MacLaurin formula established previously (80) allows us to compute the first term of this expansion, namely of order $(T/\hbar)^2$:

$$\tilde{S}(\omega_n) = \tilde{S}(\omega_n)|_{\beta\hbar=\infty} \quad (82)$$
$$+\frac{1}{(\beta\hbar)^2}\frac{2\pi}{3}\partial_x I_0(x)|_{x=0}\mathcal{J}_2(\Sigma_0)\mathcal{K}_1(\omega_n) + \mathcal{O}(\frac{1}{(\beta\hbar)^4})$$
$$= \tilde{S}(\omega_n)|_{\beta\hbar=\infty} + \left(\frac{T}{\hbar}\right)^2\frac{2\pi}{3}\int_q A_0'(0,q)\mathcal{K}_1(\omega_n)$$
$$+\mathcal{O}\left((T/\hbar)^4\right)$$

where $A_0'(\omega, q)$ is given in the text (61). We want to show that this term of order $(T/\hbar)^2$ in (82) can be obtained in a more standard way using a spectral representation to compute the sum over Matsubara frequencies. Indeed, we can write this sum (81) as

$$\tilde{S}(\omega_n) = \int_{q,q'}\int_{-\infty}^{+\infty}\frac{du_1}{\pi}\int_{-\infty}^{+\infty}\frac{du_2}{\pi}A_0(q,u_1)A_0\left(q',u_2\right)$$
$$\times\frac{1}{\beta\hbar}\sum_m\frac{1}{i\omega_m - u_1}\frac{1}{i\omega_m + i\omega_n - u_2} \quad (83)$$

with $A_0(q,\omega)$ given in (61). The sum over the Matsubara frequencies is then straightforwardly computed, and

it gives

$$\tilde{S}(\omega_n) = \int_{q,q'} \int_{-\infty}^{+\infty} \frac{du_1}{\pi} \int_{-\infty}^{+\infty} \frac{du_2}{\pi} A_0(q,u_1) A_0\left(q',u_2\right) \times \frac{f_B(u_1) - f_B(u_2)}{u_2 - u_1 - i\omega_n} \tag{84}$$

$$= \int_{q,q'} \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \frac{1}{\pi^2} A_0(q,u_1) A_0\left(q',u_2\right) \times f_B(u_1) \left( \frac{1}{u_2 - u_1 - i\omega_n} + \frac{1}{u_2 - u_1 + i\omega_n} \right). \tag{85}$$

The term in $(T/\hbar)^2$ in $\tilde{S}(\omega_n)$ is obtained by computing $\frac{\partial \tilde{S}(\omega_n)}{\partial 1/(\beta\hbar)^2}$, where the derivative concerns the explicit dependence in $\beta\hbar$, i.e. does not act on the implicit one of $\omega_n$. This leads to

$$\frac{\partial \tilde{S}(\omega_n)}{\partial 1/(\beta\hbar)^2} = \tag{86}$$

$$\frac{(\beta\hbar)^3}{2} \int_{q,q'} \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \frac{1}{\pi^2} A_0(q,u_1) A_0(q',u_2) \times \left( \frac{1}{u_2 - u_1 - i\omega_n} + \frac{1}{u_2 - u_1 + i\omega_n} \right) u_1 \frac{e^{\beta\hbar u_1}}{\left(e^{\beta\hbar u_1} - 1\right)^2}.$$

The integral over $u_1$ is well behaved due to the derivative of the Bose factor and we can safely rescale $u_1 \to \beta\hbar x_1$:

$$\frac{\partial \tilde{S}(\omega_n}{\partial 1/(\beta\hbar)^2} = \tag{87}$$

$$\frac{\beta}{2} \int_{q,q'} \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} du_2 \frac{1}{\pi^2} A_0\left(q, x_1/\beta\right) A_0\left(q',u_2\right) \times \left( \frac{1}{u_2 - x_1/\beta - i\omega_n} + \frac{1}{u_2 - x_1/\beta + i\omega_n} \right) x_1 \frac{e^{x_1}}{\left(e^{x_1} - 1\right)^2}.$$

From that expression, we extract the coefficient of order $1/(\beta\hbar)^2$:

$$\frac{\partial \tilde{S}(\omega_n)}{\partial 1/(\beta\hbar)^2}\Big|_{\beta\hbar\to\infty} = \frac{1}{2} \int_q A_0'(q,0) \int_q \int_{-\infty}^{+\infty} \frac{du_2}{\pi} A_0(q,u_2) \times \left( \frac{1}{u_2 - i\omega_n} + \frac{1}{u_2 + i\omega_n} \right) \int_{-\infty}^{+\infty} \frac{dx_1}{\pi} x_1^2 \frac{e^{x_1}}{\left(e^{x_1} - 1\right)^2} = \frac{2\pi}{3} \int_q A_0'(q,0) \mathcal{K}_1(\omega_n) \tag{88}$$

where we have used the value of the integral

$$\alpha = \int_{-\infty}^{+\infty} dx x^2 \frac{e^x}{\left(e^x - 1\right)^2} = \frac{2\pi^2}{3}. \tag{89}$$

This calculation shows explicitly the equivalence (82), (88) of the two methods to compute this low $T$ expansion. The expansion of the term entering the equation for $\Sigma_1 + I_1(\omega_n)$ can be written

$$\frac{4\hat{V}'''(0)}{(\beta\hbar)} \sum_m \mathcal{K}_1(\omega_m)(\mathcal{K}_1(\omega_m) - \mathcal{K}_1(\omega_n + \omega_n)) \tag{90}$$

$$= 4\hat{V}'''(0) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathcal{K}_1(\omega)(\mathcal{K}_1(\omega) - \mathcal{K}_1(\omega + \omega_n)) + 4\left(\frac{T}{\hbar}\right)^2 \hat{V}'''(0) \frac{2\pi}{3} \int_q A_0'(q,0)(\mathcal{J}_1(\Sigma_0) - \mathcal{K}_1(\omega_n)) + \mathcal{O}\left((T/\hbar)^4\right). \tag{91}$$

And using the equation for $I_0(\omega_n)$ given in the text (42), the term of order $(T/\hbar)^2$ in (90) can simply be written

$$\frac{4\hat{V}'''(0)}{(\beta\hbar)} \sum_m \mathcal{K}_1(\omega_m)(\mathcal{K}_1(\omega_m) - \mathcal{K}_1(\omega_n + \omega_n)) \tag{92}$$

$$= 4\hat{V}'''(0) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathcal{K}_1(\omega)(\mathcal{K}_1(\omega) - \mathcal{K}_1(\omega + \omega_n)) - I_0(\omega_n) \left(\frac{T}{\hbar}\right)^2 \frac{\hat{V}'''(0)}{\hat{V}''(0)} \frac{2\pi}{3} \int_q A_0'(q,0) + \mathcal{O}\left((T/\hbar)^4\right)$$

as given in the text (60).

## Appendix B: Low temperature expansion to order $\mathcal{O}(\hbar^2)$: detailed calculations

In this appendix, we focus on the internal energy to order $\mathcal{O}(\hbar^2)$ (66) and show how to extract the coefficient of the term $\propto 1/(\beta\hbar)^2$ in the expression

$$-\hbar^2 \frac{V'''(0)}{V''(0)} \frac{1}{2(\beta\hbar)^2} \sum_n \mathcal{K}_1(\omega_n) \sum_n I_0(\omega_n) \mathcal{K}_1(\omega_n) + \frac{2\hbar^2}{3} V'''(0) \frac{1}{(\beta\hbar)^2} \sum_{m,n} \mathcal{K}_1(\omega_m) \mathcal{K}_1(\omega_n) \mathcal{K}_1\left(\omega_n + \omega_n\right). \tag{93}$$

Let us expand the first term of (93):

$$\mathcal{I} = -\frac{V'''(0)}{V''(0)} \frac{1}{2(\beta\hbar)^2} \sum_n \mathcal{K}_1(\omega_n) \sum_n I_0(\omega_n) \mathcal{K}_1(\omega_n). \tag{94}$$

The general structure of the high $\beta\hbar$ expansion of this term is the following

$$\mathcal{I} = \mathcal{I}^{(0)} + \frac{\mathcal{I}^{(2)}}{(\beta\hbar)^2} + \mathcal{O}\left(\frac{1}{\beta^4}\right). \tag{95}$$

We are not interested in the constant as it does not contribute to the specific heat and and focus here on the computation of the first term $\mathcal{I}^{(2)}$. We use a spectral

representation to compute the Matsubara sums in $\mathcal{I}$:

$$\mathcal{K}_1(\omega_n) = \frac{1}{cq^2 + \Sigma_0 + M\omega_n^2 + I_0(\omega_n)} \tag{96}$$
$$= \frac{-1}{\pi}\int_{-\infty}^{+\infty} du A_0(q,u)\frac{1}{i\omega_n - u}$$
$$\mathcal{K}_1(\omega_n)I_0(\omega_n) = \frac{I_0(\omega_n)}{cq^2 + \Sigma_0 + M\omega_n^2 + I_0(\omega_n)} \tag{97}$$
$$= \frac{-1}{\pi}\int_{-\infty}^{+\infty} du \left(I_0''(u)B_0(q,u) + I_0'(u)A_0(q,u)\right)\frac{1}{i\omega_n - u}$$

where $A_0(q,\omega)$ is defined in the text (61) and

$$B_0(q,\omega) = \mathrm{Re}G_{c0}(i\omega_m \to \omega + i\delta) \tag{98}$$
$$= \frac{cq^2 - \omega^2 + \Sigma_0 + I_0'(\omega)}{\left(cq^2 - \omega^2 + \Sigma_0 + I_0'(u)\right)^2 + I_0''(u)^2}. \tag{99}$$

Using the identity

$$\frac{1}{\beta\hbar}\sum_n \frac{1}{i\omega_n - u} = -f_B(u). \tag{100}$$

$\mathcal{I}$ can be written

$$\mathcal{I} = \frac{-V'''(0)}{V''(0)}\frac{1}{2\pi^2}\int_{-\infty}^{+\infty} du_1 \int_q A_0(q,u_1) f_B(u_1) \tag{101}$$
$$\times \int_{-\infty}^{+\infty} du_2 \int_q \left(I_0''(u_2)B_0(q,u_2) + I_0'(u_2)A_0(q,u_2)\right) f_B(u_2).$$

We use this expression to compute the term of order $1/(\beta\hbar)^2$, together with $\frac{\partial}{\partial 1/(\beta\hbar)^2} = -\frac{(\beta\hbar)^3}{2}\frac{\partial}{\partial(\beta\hbar)}$, we have

$$\frac{\partial \mathcal{I}}{\partial 1/(\beta\hbar)^2} = \tag{102}$$
$$\frac{-V'''(0)}{V''(0)}\frac{1}{2\pi^2}\frac{(\beta\hbar)^3}{2}\int_{-\infty}^{+\infty} du_1 \int_q A_0(q,u_1)u_1 \frac{e^{\beta\hbar u_1}}{\left(e^{\beta\hbar u_1}-1\right)^2}$$
$$\times \int_{-\infty}^{+\infty} du_2 \int_q \left(I_0''(u_2)B_0(q,u_2) + I_0'(u_2)A_0(q,u_2)\right) f_B(u_2)$$
$$-\frac{V'''(0)}{V''(0)}\frac{(\beta\hbar)^3}{2}\int_{-\infty}^{+\infty} du_1 \int_q A_0(q,u_1) f_B(u_1) \int_{-\infty}^{+\infty} du_2$$
$$\times \int_q \left(I_0''(u_2)B_0(q,u_2) + I_0'(u_2)A_0(q,u_2)\right) u_2 \frac{e^{\beta\hbar u_2}}{\left(e^{\beta\hbar u_2}-1\right)^2}.$$

After some manipulations, we obtain this coefficient:

$$\frac{\partial \mathcal{I}}{\partial 1/(\beta\hbar)^2}\Big|_{\beta\hbar=\infty} = \tag{103}$$
$$\frac{V'''(0)}{V''(0)}\frac{\alpha}{4\pi^2}\Big(\int_q A_0'(0,q)\int_{-\infty}^{0} du_1$$
$$\times \int_q \left(I_0''(u_1)B_0(q,u_1) + I_0'(u_1)A_0(q,u_1)\right)$$
$$+ \int_q \left(\partial_\omega I_0''\right)(0) B_0(0,q)\int_{-\infty}^{0} du_2 \int_q A_0(q,u_2)\Big)$$

with $\alpha$ given in (89). We can simplify that expression using the equations for $I_0'(u)$ and $I_0''(u)$ obtained from (42):

$$I_0''(\omega) = 4V''(0)\int_q A_0(q,\omega) \tag{104}$$
$$I_0'(\omega) = -4V''(0)\left(\int_q \frac{1}{cq^2 + \Sigma_0} - B_0(q,\omega)\right)$$

which finally yields

$$\frac{\partial \mathcal{I}}{\partial 1/(\beta\hbar)^2}\Big|_{\beta\hbar=\infty} \tag{105}$$
$$= \frac{V'''(0)}{V''(0)}\frac{\alpha}{2\pi^2}\left(\partial_\omega I''\right)(0)\int_{q,q'}\int_{-\infty}^{0} du A_0(q,u)B_0(q',u).$$

The high $\beta\hbar$ expansion of the following term

$$\mathcal{J} = \frac{2}{3}V'''(0)\frac{1}{(\beta\hbar)^2}\sum_{m,n}\mathcal{K}_1(\omega_n)\mathcal{K}_1(\omega_n)\mathcal{K}_1(\omega_m+\omega_n)$$
$$= -\frac{2V'''(0)}{3\pi^3}\int_{-\infty}^{+\infty} du_1 du_2 du_3 A_0(u_1)A_0(u_2)A_0(u_3)$$
$$\times \frac{1}{(\beta\hbar)^2}\sum_{n,m}\frac{1}{i\omega_n - u_1}\frac{1}{i\omega_m - u_2}\frac{1}{i\omega_n + i\omega_m - u_3} \tag{106}$$

is performed using the same kind of manipulations, which lead to

$$\frac{\partial \mathcal{J}}{\partial 1/(\beta\hbar)^2}\Big|_{\beta\hbar=\infty} \tag{107}$$
$$= -\frac{V'''(0)}{V''(0)}\frac{\alpha}{2\pi^2}(\partial_\omega I_0'')(0)\int_{q,q'}\int_{-\infty}^{0} du A_0(q,u)B_0(q',u).$$

Combining (105) and (107) shows that the term of order $(T/\hbar)^2$ cancels in (93), as given in the text (69).

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